On the representation of integers as the sums of distinct summands taken from a fixed set

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To Professor L. Rédei on his 60th birthday

1. Introduction

Let $\mathfrak{A}$ be a sequence of distinct positive integers

$$a_1 < a_2 < a_3 < \cdots.$$

We write $A(n)$ for the number of elements a of $\mathfrak{A}$ with $a \leqq n$, and similarly for other sets. As usual, we put

$$\|x\| = \inf |x-n| \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

We shall establish the following theorem:

Theorem I. *Suppose that*

$$(*) \qquad \lim_{n\to\infty} \frac{A(2n)-A(n)}{\log\log n} = \infty$$

and that

$$(\ast\ast) \qquad \sum_{a\in\mathfrak{A}} \|a\theta\|^2 = \infty$$

for every real number θ in $0<\theta<1$. Then every sufficiently large number is representable as the sum of distinct elements of $\mathfrak{A}$.

We prove Theorem I in § 2 by the Hardy—Littlewood circle method. It will be seen that we do not use quite the full force of the hypotheses. By a more refined estimate of the integrals which occur, it would probably be possible to weaken the hypotheses further.

This investigation was touched off by the ingenious paper of Birch [1] about the representation of integers as the sums of the elements $p^\alpha q^\beta$ ($\alpha = 0, 1, 2, \ldots$; $\beta = 0, 1, 2, \ldots$), where p, q is a pair of coprime integers. His results are an immediate consequence of Theorem I.

In § 3 we shall prove the following result, which shows that congruence considerations and a very severe condition on the rate of increase of $A(n)$ are not alone sufficient to ensure that every sufficiently large number is the sum of distinct elements of $\mathfrak{A}$.

Theorem II. *Let $\varepsilon > 0$ be given. Then there exists a set $\mathfrak{C}$ of positive integers $c_1 < c_2 < \cdots$ with the following properties:*

(i)
$$\lim_{n \to \infty} (c_{n+1} - c_n)/c_n^{\frac{1}{2}+\varepsilon} = 0;$$

(ii) *$\mathfrak{C}$ contains infinitely many elements in every arithmetic progression;*

(iii) *if $S(n)$ denotes the number of integers $\leqq n$ which are expressible as the sum of distinct elements of $\mathfrak{C}$, then $S(n) < \varepsilon n$ for every n.*

Note that (i) implies, in particular, that

$$\lim_{n \to \infty} n^{-\frac{1}{2}+\varepsilon} C(n) = \infty.$$

On the other hand, it is easy to see that any set $\mathfrak{A}$ with

$$\liminf n^{-2/3} A(n) > 0$$

satisfies condition $(\overset{*}{*})$ of Theorem I provided that $\mathfrak{A}$ contains infinitely many elements not divisible by any given integer $m > 1$.

[*Added in proof.* Since this was written, my attention has been drawn to the paper of ROTH and SZEKERES [7]. Their point of view is rather different from mine, since they are primarily interested in the behaviour of the number of representations. From their work it follows, in particular, that condition $(*)$ may be weakened provided that condition $(\overset{*}{*})$ is appropriately strengthened. They do not require such delicate estimates of integrals as are needed in this paper.]

2. Proof of Theorem I

We shall not apply the Hardy—Littlewood method to the set $\mathfrak{A}$ directly, but to a subset $\mathfrak{B}$ of $\mathfrak{A}$. It will clearly be enough to show that every sufficiently large integer is the sum of distinct elements of $\mathfrak{B}$. The set $\mathfrak{B}$ is constructed in Lemma 1. It should perhaps be remarked that, while we have for convenience given explicit numerical values to constants occurring in the argument, we have made no attempt to make those values small and have estimated quite crudely.

Lemma 1. *There is a set* $\mathfrak{B} \subset \mathfrak{A}$ *and integers* M, N *with* $2^{40} \leqq N \leqq M$ *such that*

(i) $$\sum_{b \in \mathfrak{B},\, b \leqq 2^M} \|b\theta\|^2 > 2N + 50$$

for all real θ *in the range* $2^{-N-2} \leqq \theta \leqq 1 - 2^{-N-2}$,

(ii) $$B(2^{m+1}) - B(2^m) \geqq 2^{20} \log_2 m \quad (\text{for all} \quad m \geqq N),$$

(iii) $$B(2^{m+1}) - B(2^m) \leqq 2^{20} \log_2 m + 1 \quad (\text{for all} \quad m \geqq M),$$

where $\log_2 m = \log m / \log 2$.

First, by (i) of the enunciation of Theorem I there is an integer $N \geqq 2^{40}$ such that

(1) $$A(2n) - A(n) > 2^{20} \log_2 \log_2 n \quad (\text{for all} \quad n \geqq 2^N).$$

Secondly, by condition (ii) of Theorem I and a routine application of the Heine—Borel covering theorem, there is an integer $M \geqq N$ such that

$$\sum_{a \in \mathfrak{A},\, a \leqq 2^M} \|a\theta\|^2 > 2N + 50$$

for all θ in the range $2^{-N-2} \leqq \theta \leqq 1 - 2^{-N-2}$.

The elements b of the set $\mathfrak{B}$ in the range

$$0 < b \leqq 2^M$$

are taken to be just the elements of $\mathfrak{A}$ in that interval. The elements of $\mathfrak{B}$ in an interval

(2) $$2^m < b \leqq 2^{m+1} \qquad (m \geqq M)$$

are taken to be just any selection of

$$[2^{20} \log_2 m] + 1$$

of the elements of $\mathfrak{A}$ in that interval. (That there are so many elements of $\mathfrak{A}$ in (2) follows from (1).) The set $\mathfrak{B}$ so constructed clearly has all the required properties.

Let ϱ be a number in the range

$$0 < \varrho < 1$$

to be chosen later. The number of representations of an integer n by distinct summands from $\mathfrak{B}$ is clearly

$$\{2\pi\varrho^n\}^{-1} \int_{-\pi}^{\pi} \prod_{b \in \mathfrak{B}} (1 + \varrho^b e^{ib\theta}) e^{-in\theta} \, d\theta.$$

Hence to prove the theorem it will be enough to show that

$$\int_{-\pi}^{\pi} F(\theta) e^{-in\theta} d\theta \neq 0 \tag{3}$$

for all sufficiently large n, where

$$F(\theta) = \prod_{b \in \mathfrak{B}} \left(\frac{1 + \varrho^b e^{ib\theta}}{1 + \varrho^b} \right). \tag{4}$$

Clearly

$$|F(\theta)| \leqq 1 \tag{5}$$

for all θ and

$$F(0) = 1. \tag{6}$$

We choose ϱ, which is still at our disposal and may vary with n, so that the integrand in (3) has stationary phase in the neighbourhood of $\theta = 0$, that is so that

$$n = \sum_{b \in \mathfrak{B}} \frac{b \varrho^b}{1 + \varrho^b}. \tag{7}$$

This is possible since the right hand side of (7) increases continuously from 0 to ∞ as ϱ increases from 0 to $1-0$. The rest of this § 2 is devoted to showing that (3) holds provided that

$$n > \max \left\{ 2^{10M}, 2^{2^{400}} \right\}. \tag{8}$$

First we require an estimate of ϱ. It is convenient to work with the integer λ defined by the inequalities

$$\varrho^{2^\lambda} \geqq \frac{1}{2} > \varrho^{2^{\lambda+1}}. \tag{9}$$

From now on, n satisfies (8), and ϱ, λ are given by (7), (9) respectively.

Lemma 2.

$$\lambda \geqq \max \{ 8M, 2^{2^{300}} \}. \tag{10}$$

For, by (7) we have

$$n \leqq \sum_{b \in \mathfrak{B}} b \varrho^b \leqq \sum_{\substack{b \in \mathfrak{B} \\ b \leqq 2^M}} b + \sum_{m \geqq M} \sum_{\substack{2^m < b \leqq 2^{m+1} \\ b \in \mathfrak{B}}} b \varrho^b \leqq$$

$$\leqq 2^{2M} + \sum_{m \geqq 1} (2^{20} \log_2 m + 1) 2^{m+1} \varrho^{2^m} \leqq \frac{1}{2} n + \sum_{1 \leqq m \leqq \lambda} (2^{20} \log_2 m + 1) 2^{m+1} + \tag{11}$$

$$+ \sum_{m > \lambda} (2^{20} \log_2 m + 1) 2^{m+1} (1/2)^{2^{m-\lambda+1}} \leqq \frac{1}{2} n + (2^{20} \log_2 \lambda + 1) 2^{\lambda+5}.$$

The Lemma now follows from (8) and (11).

It is convenient to state for further reference two inequalities involving ϱ and λ. We do not give the proofs, which are all by a division of the range of summation as in the proof of Lemma 2.

Corollary. *The following estimates hold:*

$$\sum_{b\in\mathfrak{B}} b^3\varrho^b < 2^{3\lambda+30}\log_2\lambda, \tag{12}$$

$$\sum_{b\in\mathfrak{B}} \frac{b^2\varrho^b}{(1+\varrho^b)^2} < 2^{2\lambda+30}\log_2\lambda. \tag{13}$$

Lemma 3. *Let σ, Φ be real numbers such that*

$$\frac{1}{2}\leqq\sigma\leqq 1,\quad |\Phi|\leqq\pi. \tag{14}$$

Then

$$\left|\frac{1+\sigma e^{i\Phi}}{1+\sigma}\right|^2\leqq\exp\{-4\,\Phi^2/9\pi^2\}\leqq 2^{-\Phi^2/2\pi^2}. \tag{15}$$

For

$$|1+\sigma e^{i\Phi}|^2=(1+\sigma)^2-2\sigma\sin^2\Phi/2$$

and

$$|\sin\Phi/2|\geqq|\Phi|/\pi.$$

Hence

$$\left|\frac{1+\sigma e^{i\Phi}}{1+\sigma}\right|^2\leqq 1-\frac{2\sigma}{(1+\sigma)^2}\frac{\Phi^2}{\pi^2}\leq 1-4\,\Phi^2/9\pi^2\leqq\exp\{-4\,\Phi^2/9\pi^2\}.$$

Corollary 1. *Suppose that*

$$2^{-N-1}\pi\leqq\theta\leqq(2-2^{-N-1})\pi. \tag{16}$$

Then

$$|F(\theta)|\leqq 2^{\,2N-50}, \tag{17}$$

where $F(\theta)$ is given by (4).

For

$$|F(\theta)|\leqq\prod_{b\in\mathfrak{B},\,b\leqq 2^M}\left|\frac{1+\varrho^b e^{ib\theta}}{1+\varrho^b}\right|.$$

In each of the terms of the product we have

$$1\geqq\varrho^b\geqq\varrho^{2^M}\geqq\varrho^{2^\lambda}\geqq\frac{1}{2}$$

by (9). Hence on applying Lemma 3 to each term, with

$$\sigma=\varrho^b\quad\text{and}\quad\Phi=\pm\,2\pi\|b\theta/2\pi\|,$$

we have

$$|F(\theta)| \leqq 2^{-\Sigma\|b\theta/2\pi\|^2},$$

where the sum is over $b \in \mathfrak{B}$, $b \leqq 2^M$. The truth of (17) now follows from Lemma 1 (i).

For

$$N \leqq m < \lambda$$

we write

$$f_m(\theta) = \prod_{\substack{b \in \mathfrak{B} \\ 2^m < b \leqq 2^{m+1}}} \left(\frac{1 + \varrho^b e^{ib\theta}}{1 + \varrho^b} \right). \tag{18}$$

Corollary 2.

$$|f_{\lambda-1}(\theta)| \leqq \exp\{-2^{2\lambda+10}(\log_2 \lambda)\theta^2\} \tag{19}$$

for all θ in the range $|\theta| \leqq 2^{-\lambda}\pi$.

By Lemma 3 we have the estimate

$$|f_{\lambda-1}(\theta)|^2 \leqq \exp\left\{ \frac{-4}{9\pi^2} \sum_{\substack{b \in \mathfrak{B} \\ 2^{\lambda-1} < b \leqq 2^\lambda}} (b\theta)^2 \right\}.$$

Since there are at least $2^{20}\log_2(\lambda - 1)$ summands, by Lemma 1 (ii), the required estimate follows.

Corollary 3. *Let*

$$\theta_0 = (\log_2 \lambda)^{-2/5} 2^{-\lambda}\pi. \tag{20}$$

Then

$$\int_{\theta_0}^{2^{-\lambda}\pi} |F(\theta)|\,d\theta \leqq 2^{-\lambda}(\log_2 \lambda)^{-1/2} \exp\{-2^{10}(\log_2 \lambda)^{1/5}\}. \tag{21}$$

For

$$|F(\theta)| \leqq |f_{\lambda-1}(\theta)| \tag{22}$$

and Corollary 3 follows from Corollary 2 with a little calculation.

Lemma 4. *Let σ, Φ, ψ be real numbers such that*

$$\frac{1}{2} \leqq \sigma \leqq 1, \quad \frac{\pi}{8} \leqq |\Phi - \psi| \leqq \pi. \tag{23}$$

Then

$$\left| \frac{1 + \sigma e^{i\Phi}}{1 + \sigma} \right| \left| \frac{1 + \sigma e^{i\psi}}{1 + \sigma} \right| \leqq 2^{-2^{-9}}. \tag{24}$$

Since

$$\frac{d}{d\Phi}\log|1+\sigma e^{i\Phi}|=\frac{-\sigma\sin\Phi}{1+\sigma^2+2\sigma\cos\Phi}$$

is monotone in $|\Phi|\leqq\pi/2$, it is easy to see that the left hand side of (26) attains its maximum when $\Phi=-\psi=\pi/16$ for fixed σ, when Φ,ψ are allowed to vary. Lemma 4 now follows from Lemma 3.

Corollary 1. *Let $N\leqq m<\lambda$ and let θ_0,θ_1 be two numbers such that*

$$2^{-m-3}\pi\leqq|\theta_0-\theta_1|\leqq 2^{-m-1}\pi. \tag{25}$$

Then

$$|f_m(\theta_0)f_m(\theta_1)|\leqq m^{-10}, \tag{26}$$

where $f_m(\theta)$ is defined in (18).

For, by Lemma 1 (ii),

$$|f_m(\theta_0)f_m(\theta_1)|=\prod_{\substack{2^m<b\leqq 2^{m+1}\\ b\in\mathfrak{B}}}\left|\frac{1+\varrho^b e^{ib\theta_0}}{1+\varrho^b}\right|\left|\frac{1+\varrho^b e^{ib\theta_1}}{1+\varrho^b}\right|$$

is the product of at least $2^{20}\log_2 m$ terms to each of which Lemma 4 applies.

Corollary 2. *Let $\mathfrak{J}$ be any interval of length*

$$|\mathfrak{J}|\leqq 2^{-m-1}\pi. \tag{27}$$

Then $\mathfrak{J}$ can be divided into 3 intervals $\mathfrak{K},\mathfrak{L},\mathfrak{M}$ each of length at most $2^{-m-2}\pi$, so that[1])

$$|f(\theta)|\leqq\frac{1}{4}m^{-4}\qquad(\theta\in\mathfrak{K},\mathfrak{L}). \tag{28}$$

We may suppose that $\mathfrak{J}$ is closed, so that $|f_m(\theta)|$ for $\theta\in\mathfrak{J}$ takes its maximum at some $\theta_0\in\mathfrak{J}$. If $|f_m(\theta_0)|\leqq\frac{1}{4}m^{-4}$, there is nothing to prove. If $|f_m(\theta_0)|>\frac{1}{4}m^{-4}$ we take for $\mathfrak{M}$ the points of $\mathfrak{J}$ for which $|\theta-\theta_0|\leqq 2^{-m-3}\pi$. By corollary 1 we have

$$|f_m(\theta)|\leqq m^{-10}/|f_m(\theta_0)|\leqq 4m^{-6}\leqq\frac{1}{4}m^{-4}$$

for $\theta\in\mathfrak{J},\theta\notin\mathfrak{M}$; and the result follows.

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1) We do not exclude the possibility that one of the intervals is empty.

L e m m a 5. *Let $\mathfrak{J}$ be any interval of length $|\mathfrak{J}| \leqq 2^{-\lambda}\pi$. Then*

$$\int_{\mathfrak{J}} |f_{\lambda-1}(\theta)|^{1/2} d\theta \leqq (\log_2 \lambda)^{-1/2} 2^{-\lambda}. \tag{29}$$

We use the rearrangement of functions as explained in HARDY, LITTLEWOOD and PÓLYA [5] Chapter X. If $u(\theta)$ is any continuous positive function defined for $\theta \in \mathfrak{J}$, then, just for this proof, we denote by $u^*(\theta)$ the symmetric rearrangement of $u(\theta)$; that is the unique function defined for $|\theta| \leqq \frac{1}{2}|\mathfrak{J}|$ by the following properties:

(A) $u^*(\theta)$ is continuous,

(B) $u^*(-\theta) = u^*(\theta)$,

(C) $u^*(\theta)$ decreases for $0 \leqq \theta \leqq \frac{1}{2}|\mathfrak{J}|$,

(D) for any number u_0, the set of θ in $|\theta| \leqq \frac{1}{2}|\mathfrak{J}|$ such that $u^*(\theta) > u_0$ has the same measure as the set of $\theta \in \mathfrak{J}$ such that $u(\theta) > u_0$.

If $u(\theta), v(\theta)$ are any two positive continuous functions defined in $\mathfrak{J}$ then it is easy to see that

$$\int_{\mathfrak{J}} u(\theta)v(\theta)\,d\theta \leqq \int_{|\theta| \leqq \frac{1}{2}|\mathfrak{J}|} u^*(\theta)v^*(\theta)\,d\theta. \tag{30}$$

(cf. HARDY, LITTLEWOOD and PÓLYA [5], Theorem 378).

In particular, when $\mathfrak{J}$ is as above and

$$u(\theta) = u_b(\theta) = \left| \frac{1+\varrho^b e^{i\theta}}{1+\varrho^b} \right|^{1/2}$$

for some integer b with $b \leqq 2^\lambda$, it is obvious that

$$u^*(\theta) \leqq u(\theta) \qquad \left(|\theta| \leqq \frac{1}{2}|\mathfrak{J}|\right).$$

Since

$$|f_{\lambda-1}(\theta)|^{1/2} = \prod_{\substack{b \in \mathfrak{B} \\ 2^{\lambda-1} < b \leqq 2^\lambda}} u_b(\theta),$$

repeated application of (30) gives

$$\int_{\mathfrak{J}} |f_{\lambda-1}(\theta)|^{1/2} d\theta \leqq \int_{|\theta| \leqq 2^{-\lambda-1}\pi} |f_{\lambda-1}(\theta)|^{1/2} d\theta \leqq \int_{-\infty<\theta<\infty} \exp\{-2^{2\lambda+9}(\log_2 \lambda)\theta^2\}\,d\theta,$$

by Lemma 3, Corollary 2.

Lemma 6. *Let m be any integer in $N \leqq m < \lambda$ and let $\mathfrak{I}$ be any interval of length $|\mathfrak{I}| \leqq 2^{-m-1}\pi$. Then*

$$S_m(\mathfrak{I})\ (\text{say}) = \int_{\mathfrak{I}} \Big| \prod_{m \leqq \mu < \lambda} f_\mu(\theta) \Big|^{1/2} d\theta < (\log_2 \lambda)^{-1/2} 2^{-\lambda} \prod_{m \leqq \mu < \lambda} (1+\mu^{-2}).$$

When $m = \lambda - 1$, this is just Lemma 5. Otherwise we use backwards induction on m. Let $\mathfrak{K}, \mathfrak{L}, \mathfrak{M}$ be the intervals given by Lemma 4, Corollary 4. Then, in an obvious notation,

$$S_m(\mathfrak{I}) = S_m(\mathfrak{K}) + S_m(\mathfrak{L}) + S_m(\mathfrak{M}) \leqq$$

$$\leqq \frac{1}{2} m^{-2} S_{m+1}(\mathfrak{K}) + \frac{1}{2} m^{-2} S_{m+1}(\mathfrak{L}) + S_{m+1}(\mathfrak{M}),$$

since $|f_m(\theta)|^{1/2} \leqq \frac{1}{2} m^{-2}$ in $\mathfrak{K}, \mathfrak{L}$ and $|f_m(\theta)|^{1/2} \leqq 1$ in $\mathfrak{M}$. The intervals $\mathfrak{K}, \mathfrak{L}, \mathfrak{M}$ satisfy the condition of the Lemma with $m+1$ instead of m and the result follows.

Corollary. *Let $N \leqq m \leqq \lambda - 2$ and let $\mathfrak{I}^m$ be the interval*

$$2^{-\lambda}\pi \leqq \theta \leqq 2^{-m-1}\pi.$$

Then

$$S_m(\mathfrak{I}^m) \leqq (\log_2 \lambda)^{-1/2} 2^{-\lambda} \Big\{ \prod_{m \leqq \mu < \lambda} (1+\mu^{-2}) - 1 \Big\}. \tag{31}$$

For $\mathfrak{I}^m$ is the union of $\mathfrak{I}^{m+1}$ and $\mathfrak{K}^{m+1}$, where $\mathfrak{K}^{m+1}$ is the interval

$$2^{-m-2}\pi < \theta \leqq 2^{-m-1}\pi.$$

By Lemma 4, Corollary 1, we have

$$|f_m(\theta)| \leqq m^{-10} < m^{-2}$$

for $\theta \in \mathfrak{K}^{m+1}$, since $f_m(0) = 0$. The required estimate now follows by induction on applying Lemma 6 with $m+1$ for m to the interval $\mathfrak{K}^{m+1}$ instead of $\mathfrak{I}$, and since

$$S_m(\mathfrak{I}^m) = S_m(\mathfrak{I}^{m+1}) + S_m(\mathfrak{K}^{m+1}) \leqq S_{m+1}(\mathfrak{I}^{m+1}) + m^{-2} S_m(\mathfrak{K}^{m+1}).$$

Lemma 7. *Let*

$$\theta_0 = (\log_2 \lambda)^{-2/5} 2^{-\lambda} \pi. \tag{32}$$

Then

$$\int_{\theta_0}^{2\pi - \theta_0} |F(\theta)|\, d\theta < 2^{-\lambda-20} (\log_2 \lambda)^{-1/2}. \tag{33}$$

The integral in question is twice the integral over the range $\theta_0 \leqq \theta \leqq \pi$. We divide this into $2^{N+1}+1$ subintervals, namely

$\mathfrak{K}$: $\quad \theta_0 \leqq \theta \leqq 2^{-\lambda}\pi$,

$\mathfrak{L}$: $\quad 2^{-\lambda}\pi \leqq \theta \leqq 2^{-N-1}\pi$,

$\mathfrak{M}_j$: $\quad j2^{-N-1}\pi \leqq \theta \leqq (j+1)2^{-N-1}\pi \qquad (1 \leqq j \leqq 2^{N+1})$.

Lemma 3, Corollary 3 gives at once

$$\int_{\mathfrak{K}} \leqq (\log_2 \lambda)^{-1/2} 2^{-\lambda} \exp\{-2^{10}(\log_2 \lambda)^{1/5}\} < 2^{-\lambda-25}(\log_2 \lambda)^{-1/2}, \tag{34}$$

by (10).

In the interval $\mathfrak{L}$ we have, by (4) and (18),

$$|F(\theta)| \leqq \prod_{N \leqq \mu < \lambda} |f_\mu(\theta)| \leqq \prod_{N \leqq \mu < \lambda} |f_\mu(\theta)|^{1/2}.$$

Hence Lemma 6, Corollary with $m = N$ gives

$$\int_{\mathfrak{L}} \leqq (\log_2 \lambda)^{-1/2} 2^{-\lambda} \{\prod_{N \leqq \mu < \lambda} (1+\mu^{-2}) - 1\} \leqq 2^{-\lambda-25}(\log_2 \lambda)^{-1/2}. \tag{35}$$

Finally, by Lemma 3, Corollary 1, in the intervals $\mathfrak{M}_j$ we have

$$F(\theta) \leqq 2^{-2N-50}$$

and so

$$|F(\theta)| \leqq 2^{-N-25} |F(\theta)|^{1/2} \leqq 2^{-N-25} \prod_{N \leqq \mu < \lambda} |f_\mu(\theta)|^{1/2}. \tag{36}$$

Hence by Lemma 6 with $m = N$ we have

$$\int_{\mathfrak{M}_j} \leqq 2^{-N-25} (\log_2 \lambda)^{-1/2} 2^{-\lambda} \prod_{N \leqq \mu < \lambda} (1+\mu^{-2}) \leqq 2^{-\lambda-N-24}(\log_2 \lambda)^{-1/2}. \tag{37}$$

The truth of the Lemma now follows at once from (34), (35) and (37).

Lemma 8. *Let*

$$\theta_0 = (\log_2 \lambda)^{-2/5} 2^{-\lambda}\pi. \tag{38}$$

Then

$$\Re \int_{-\theta_0}^{\theta_0} F(\theta) e^{-in\theta} d\theta > 2^{-\lambda-20}(\log_2 \lambda)^{-1/2}. \tag{39}$$

Write

$$v_b(\theta) = \log \left\{ \frac{1+\varrho^b e^{ib\theta}}{1+\varrho^b} \right\}. \tag{40}$$

Then

$$v_b'''(\theta) = \frac{-\varrho^b(1-\varrho^b e^{ib\theta})b^3}{(1+\varrho^b e^{ib\theta})^3}.$$

If now

$$|\theta| \leqq \theta_0,$$

one readily obtains the estimate

$$(41) \qquad |v_b'''(\theta)| \leqq 6\varrho^b b^3,$$

on considering separately the two cases

$$b \geqq 2^{\lambda+1}, \quad \text{so} \quad \varrho^b < \frac{1}{2}$$

and

$$b < 2^{\lambda+1}, \quad \text{so} \quad |b\theta| \leqq 2\pi(\log_2\lambda)^{-2/5} \leqq \pi/4,$$

by Lemma 2. Hence, using TAYLOR's theorem with remainder, we have

$$(42) \qquad \left|\log\left(\frac{1+\varrho^b e^{ib\theta}}{1+\varrho^b}\right) - \frac{i\varrho^b b\theta}{1+\varrho^b} + \frac{\varrho^b(b\theta)^2}{2(1+\varrho^b)^2}\right| \leqq \varrho^b|b\theta|^3 \leqq \varrho^b(b\theta_0)^3.$$

Summing over $b \in \mathfrak{B}$ and recollecting that ϱ was chosen so that

$$n = \sum_{b\in\mathfrak{B}} \frac{b\varrho^b}{1+\varrho^b}$$

we have

$$(43) \qquad |\log\{F(\theta)e^{-in\theta}\} + \tau\theta^2| \leqq \sum_{b\in\mathfrak{B}} \varrho^b(b\theta_0)^3,$$

where

$$(44) \qquad 2\tau = \sum_{b\in\mathfrak{B}} \frac{b^2\varrho^b}{(1+\varrho^b)^2}.$$

The sums occurring here were estimated in Lemma 2, Corollary. In the first place,

$$(45) \qquad \Sigma\varrho^b(b\theta_0)^3 \leqq 2^{3\lambda+30}(\log_2\lambda)\theta_0^3 \leqq \pi^3 2^{30}(\log_2\lambda)^{-1/5} < 2^{-4}$$

by Lemma 2. Further, by Lemma 2, Corollary and (44),

$$(46) \qquad 2\tau < 2^{2\lambda+30}\log_2\lambda = 2\tau_2 \text{ (say)}.$$

By (43), (45), (46) it follows that

$$(47) \quad \Re\int_{-\theta_0}^{\theta_0} F(\theta)e^{-in\theta}d\theta \geqq \frac{1}{2}\int_{-\theta_0}^{\theta_0}\exp\{-\tau\theta^2\}d\theta \geqq \frac{1}{2}\int_{-\theta_0}^{\theta_0}\exp\{-\tau_0\theta^2\}d\theta \geqq \frac{1}{4}\tau_0^{-1/2},$$

since $\tau_0\theta_0^2 > 1$ by (38) and (46).

The truth of the Lemma now follows from (46) and (47).

We note now that Lemmas 7, 8 together show that

$$\Re \int_{-\pi}^{\pi} F(\theta) e^{-in\theta} d\theta > 0.$$

This implies (3), and so the truth of the Theorem, as was remarked in the discussion that led up to (3).

3. Proof of Theorem II

Let α be any irrational number with bounded partial quotients and let $\varepsilon > 0$ be arbitrarily small. We consider first the set $\mathfrak{D}$ of positive integers

$$d_1 < d_2 < \cdots$$

such that

$$\|d\alpha\| < d^{-\frac{1}{2}(1+\varepsilon)},$$

Lemma 9. *There are numbers $\gamma_1 > 0, \gamma_2$ depending only on α and ε, such that*

$$\gamma_1 < \frac{d_{j+1} - d_j}{d_j^{\frac{1}{2}(1+\varepsilon)}} < \gamma_2,$$

for all sufficiently large j.

We have

$$\|(d_{j+1} - d_j)\alpha\| \leqq \|d_{j+1}\alpha\| + \|d_j\alpha\| < 2d_j^{-\frac{1}{2}(1+\varepsilon)}.$$

This gives the left-hand inequality, since α has bounded partial quotients. (See, for example, Cassels [3], Chapter I.)

Now let d_j be given and let $p'/q', p''/q'', p'''/q'''$ be the three consecutive best approximations to α such that

$$q' \leqq 4d_j^{\frac{1}{2}(1+\varepsilon)} < q'' < q'''.$$

Then

$$q' < q''' < \gamma_3 d_j^{\frac{1}{2}(1+\varepsilon)},$$

where γ_3 depends only on α, and

$$\|q''\alpha\| = |q''\alpha - p''| < p''^{-1} < \frac{1}{4} d_j^{-\frac{1}{2}(1+\varepsilon)}, \qquad \|q'''\alpha\| = |q'''\alpha - q'''| < \frac{1}{4} d_j^{-\frac{1}{2}(1+\varepsilon)}.$$

Further, $(q''\alpha - p'')$ and $(q'''\alpha - p''')$ have opposite signs, so

$$\min\{\|(d_j + q'')\alpha\|, \|(d_j + q''')\alpha\|\} \leqq \frac{3}{4} d_j^{-\frac{1}{2}(1+\varepsilon)}.$$

Hence at least one of the two numbers d_j+q'' and d_j+q''' lies in $\mathfrak{D}$ provided that d_j is large enough. This proves the right-hand inequality.

Corollary 1.

(48) $$\lim_{j\to\infty}\frac{d_{j+1}-d_j}{d_j^{1/2+\varepsilon}}=0.$$

Corollary 2.

(49) $$\sum_{d\in\mathfrak{D}}\|d\alpha\|<\infty.$$

Corollary 1 is immediate. To prove Corollary 2, we note that, by Lemma 9, there are $O\left(2^{\frac{1}{2}j(1+\varepsilon)}\right)$ elements $d\in\mathfrak{D}$ in $2^j<d\leqq 2^{j+1}$. Hence the contribution of the terms with $2^j<d\leqq 2^{j+1}$ to the sum in (49) is $O\left(2^{-\frac{1}{2}j\varepsilon}\right)$. Since $\sum_j 2^{-\frac{1}{2}j\varepsilon}<\infty$, the result follows.

We also consider the set $\mathfrak{T}$ of integers $t>0$ such that

$$\|t\alpha\|<\frac{1}{4}\varepsilon.$$

Since α is irrational, the fractional parts of $n\alpha$ $(n=1,2,3,\ldots)$ are uniformly distributed, and hence

$$\lim_{n\to\infty}n^{-1}T(n)=\frac{1}{2}\varepsilon$$

(see, for example, CASSELS [3], Chapter IV).

Hence, and by Lemma 9, Corollary 2, there exists an integer N such that

(50) $$T(n)<\varepsilon n \qquad (\text{for all} \quad n\geqq N)$$

and

$$\sum_{d\in\mathfrak{D},\, d\geqq N}\|d\alpha\|<\frac{1}{4}\varepsilon.$$

Denote by $\mathfrak{C}$ the set of elements c of $\mathfrak{D}$ with $c\geqq N$. If s is the sum of distinct elements of $\mathfrak{C}$, we have clearly

(51) $$s\geqq N$$

and

$$\|s\alpha\|\leqq\sum_{c\in\mathfrak{C}}\|c\alpha\|<\frac{1}{4}\varepsilon,$$

that is,

(52) $$s\in\mathfrak{T}.$$

The set $\mathfrak{C}$ has the property (i) of the enunciation of Theorem II by Lemma 9, Corollary 1, and it has property (iii) by (50), (51) and (52). It remains only to show that $\mathfrak{C}$ contains infinitely many elements in every arithmetic progression, say in $lu+v$ $(l=0,1,2,\ldots)$, where $u>0, v>0$ are fixed integers. That is, we must show that there exist infinitely many integers l such that

$$lu+v \geqq N,$$

(53)
$$\|(lu+v)\alpha\| \leqq (lu+v)^{-\frac{1}{2}(1+\delta)}.$$

But now an old theorem of KHINTCHINE [6] states that if θ is irrational and β is any real number, then

(54)
$$\liminf_{l\to\infty} l\|l\theta+\beta\| \leqq 5^{-1/2}.$$

This theorem with $\theta=u\alpha$, $\beta=v\alpha$ certainly implies the existence of infinitely many solutions of (53). (For a short proof of a slightly stronger form of KHINTCHINE's theorem see CASSELS [2]. For the latest in this direction see DESCOMBES [4].)

References

[1] B. J. BIRCH, On a problem of Erdős, *Proceedings Cambridge Phil. Soc.*, **55** (1959), 370—373.

[2] J. W. S. CASSELS, The lattice properties of asymmetric hyperbolic regions. I. On a theorem of Khintchine, *Proceedings Cambridge Phil. Soc.*, **44** (1948), 1—7.

[3] J. W. S. CASSELS, *An introduction to Diophantine approximation*, Cambridge Tract N° 45 (Cambridge, 1957).

[4] R. DESCOMBES, Sur la répartition des sommets d'une ligne polygonale régulière non fermée, *Annales École Normale Supérieure*, **73** (1956), 283—355.

[5] G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, *Inequalities* (Cambridge, 1934).

[6] A. KHINTCHINE, Neuer Beweis und Verallgemeinerung eines Hurwitzschen Satzes, *Math. Annalen*, **111** (1935), 631—637.

[7] K. F. ROTH—G. SZEKERES, Some asymptotic formulae in the theory of partitions, *Quarterly J. Math.*, **5** (1954), 241—259.

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