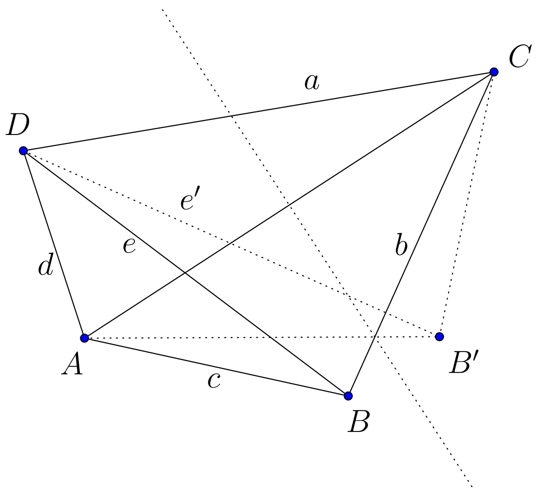
The present short mathematical note is devoted to the analysis of inequalities of the type

$$e+f \leq L(a,b,c,d) := xa+yb+zc+td$$

with $x,y,z,t \geq 0$ and that should hold for any quadrilateral $ABCD$ with side and diagonal lengths $a \geq b \geq c \geq d$ and e,f respectively.

Note 1

In proving a generic estimate of the type $e+f \leq L(a,b,c,d)$ that should hold for an arbitrary quadrilateral $ABCD$ with side and diagonal lengths $a \geq b \geq c \geq d$ and e,f respectively one can assume w.l.o.g. that $ABCD$ is convex and also that the lengths a and b correspond to opposite sides. To prove for example the latter claim assume the sides labeled a and b are adjacent, as shown in figure below. We deduce that B,D lie on the same side of the bisector line of the segment AC. Considering B' to be the reflection of B in this bisector line we have that $e=|DB| \leq |DB'| = e'$ whereas the quadrilaterals $ABCD$ and $AB'CD$ have the same four side lengths. It follows that if we are able to prove $e'+f \leq L(a,b,c,d)$ then $e+f \leq L(a,b,c,d)$ follows too.



It can be similarly shown that w.l.o.g. one can assume that $ABCD$ is a convex quadrilateral, i.e. A,C are on different sides of the line BD and also B,D are on different sides of the line AC. Supposing for example that A,C lie on the same side of the line BD, considering C' obtained by mirroring C in the line BD we deduce that $ABC''D$ is, in the sense of the inequality to be shown, a "worse" quadrilateral than $ABCD$ since $|AC'| \geq |AC|$ and $|BC| = |BC'|, |DC| = |DC'|$. Repeating this construction several times we reach in a finite number of steps a convex quadrilateral.

Problem 1
Let $ABCD$ be a quadrilateral whose side lengths are $a \geq b \geq c \geq d$. If e, f denote the lengths of the two diagonals, then

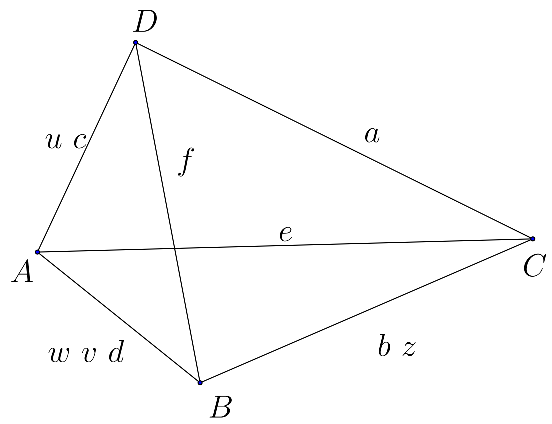
$$e + f \leq a + b + d. \tag{1}$$

Solution
The main ingredients of our argumentation are the basic triangle inequality and the Ptolemy inequality. We distinguish the following two cases.

1. $\max(e, f) \leq a$,
2. $\max(e, f) > a$.

Note that in the following we will use a, b, c, d, e, f not only to denote side lengths but also as labels of the corresponding sides and diagonals of the generic quadrilateral $ABCD$.

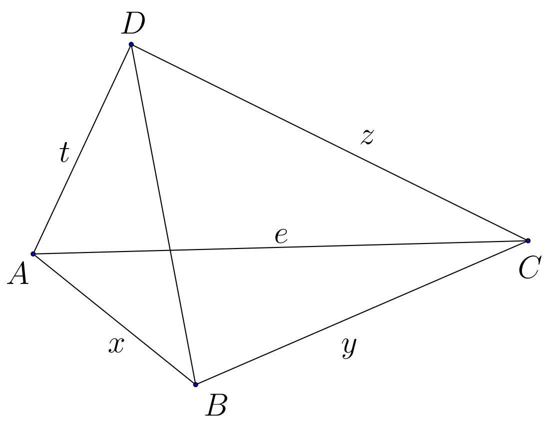
1. W.l.o.g. we assume that $f \leq e \leq a$ and show that this leads to the desired estimate (1). We consider two adjacent sides labeled $u, v \in \{b, c, d\}$ and that are building a triangle together with f. We thus have $f \leq u+v$ hence $e+f \leq a+u+v$ too. We visualize the construction in the figure below. Note that here we represent $ABCD$ as a convex quadrilateral but the convexity property is not used in our proof which is fact valid for any set of four points A, B, C, D in the plane.



On the other hand $f \leq e \leq a$ and considering two sides labeled $w, z \in \{b, c, d\}$ building a triangle together with e we deduce $e \leq w + z$ hence $e + f \leq a + w + z$.

The two pairs (u, v) and (w, z) are different (as they are building triangles together with f and e respectively) and we have thus identified two different sums of three side lengths ($a + u + v$ and $a + w + z$) bounding $e + f$ from above. As only $a + b + c$ is a sum of three side lengths that can exceed $a + b + d$, it follows that $a + b + d$ represents an upper bound of $e + f$ too and the proof of (1) is thus complete in the case $e \leq a$. $\diamond$

2. W.l.o.g. we can assume that $e = |AC|$ is the length of the longest diagonal so that $e > a$. Denote now by x, y, z, t the lengths of the four sides of $ABCD$, in consecutive order: $x = |AB|, y = |BC|, z = |CD|, t = |DA|$. It follows that x, y, z, t represent a permutation of a, b, c, d (recall $a \geq b \geq c \geq d$) and the pairs of labels (x, z) and (y, t) correspond to opposite sides too, as shown in the figure below.



By the Ptolemy inequality we have

$$ef \leq xz + yt. \tag{2}$$

Introducing for convenience the notation $K := xz + yt$ we first note that $a \geq b \geq c \geq d$ ensures

$$0 \leq K \leq ab + cd$$

and from (2) we immediately deduce

$$e + f \leq e + K/e.$$

Due to $K \geq 0$ the mapping F defined by

$$F: \quad \mathbb{R}_+ \ni u \mapsto u + K/u \in \mathbb{R}$$

is convex hence when restricted to an interval I it attains its maximum at the boundary points of I. With this in mind we note that, by the triangle inequality and the Case 2 assumption $e \geq a$, we trivially have $a \leq e \leq \min(x+y, z+t)$ hence

$$e + f \leq \max(F(a), \min(F(x+y), F(z+t))). \tag{3}$$

The three evaluations of F on the r.h.s. of the inequality above can be further estimated from above as follows.

$$\begin{aligned} F(a) &= a + K/a \leq a + (ab+cd)/a \leq a + b + cd/a \leq a + b + d \quad (4) \\ F(x+y) &= x + y + (xz+yt)/(x+y) \leq x + y + \max(z,t) \quad (5) \\ F(z+t) &= z + t + (xz+yt)/(z+t) \leq z + t + \max(x,y). \quad (6) \end{aligned}$$

The r.h.s. of (5), (6) above represent two different sums of three side lengths each, so at least one of them does not exceed $a + b + d$ (as $a + b + c$ is the only sum of three side lengths that can exceed $a + b + d$). We have thus obtained

$$\min(F(x+y), F(z+t)) \leq a + b + d. \tag{7}$$

The desired conclusion (1) follows now directly from (3), (4), (7).

Finally we note that from the proof it follows that the equality in (1) is attained exactly for $ABCD$ degenerate to the triangle ABC with $D = A$ and $|AB| = a, |AC| = b$. $\diamond$

Problem 2

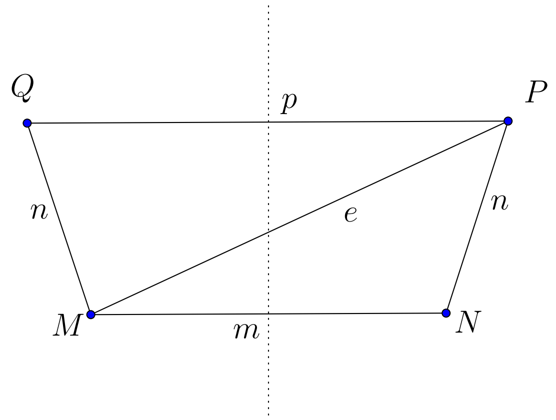
Let $ABCD$ be a quadrilateral whose side and diagonal lengths are $a \geq b \geq c \geq d$ and e, f respectively. Show that

$$e + f \leq a/2 + 3b/2 + c/2 + d/2. \tag{8}$$

Solution

We first prove the following auxiliary result. Let $MNPQ$ be an isosceles trapezoid with $MN \parallel PQ$. If $m := |MN|$, $n := |NP|$, $p := |PQ|$, $e := |MP|$ and $m \geq n$ then

$$e \leq p/4 + 3m/4 + n/2.$$



To prove this, note first that $MNPQ$ is an cyclic quadrilateral, hence by the Ptolemy equality

$$e^2 = mp + n^2.$$

The inequality to prove, under the assumptions $m \geq n$ and $|m - p| \leq 2n$, thus reads

$$mp + n^2 \leq (p/4 + 3m/4 + n/2)^2$$

which after elementary algebraic manipulations is found to be equivalent to

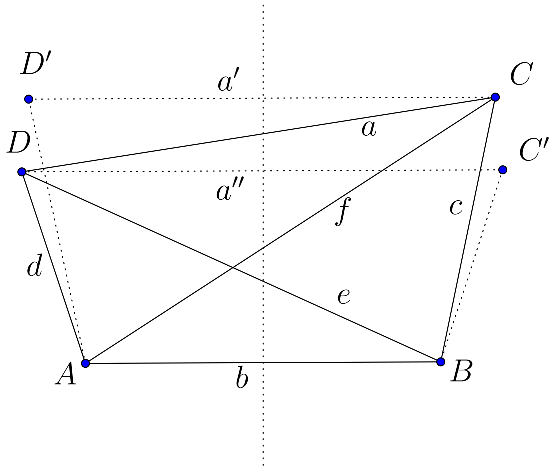
$$(p - (9m - 6n))(p - (m + 2n)) \geq 0.$$

The latter estimate holds due to $9m - 6n \geq m + 2n \geq p$, immediate consequences of $m \geq n$ and $p - m \leq 2n$.

Consider now a generic quadrilateral $ABCD$ and the isosceles trapezoids $ABC'D$ and $ABCD'$ constructed as shown in the figure below. Note that w.l.o.g. we can assume C, D to be on different sides of the bisector line of the segment AB, otherwise $e \leq d$ or $f \leq c$ and in these cases the desired conclusion follows from the triangle inequalities $f \leq b + c$ and $e \leq d + b$ respectively:

$$e \leq d: \quad e + f \leq d + b + c \leq a/2 + 3b/2 + c/2 + d/2$$

$$f \leq c: \quad e + f \leq d + b + c \leq a/2 + 3b/2 + c/2 + d/2.$$



Since $b \geq c, d$ the previous result concerning isosceles trapezoids can be applied twice to deduce

$$e \leq a''/4 + 3b/4 + d/2 \quad \text{and} \quad f \leq a'/4 + 3b/4 + c/2$$

so that

$$e + f \leq (a' + a'')/4 + 3b/2 + c/2 + d/2$$

and the conclusion follows by noting that the triangle inequality in the isosceles trapezoid $DC'CD'$ implies $a' + a'' \leq |C'D'| + |CD| = 2a$ (recall C, D are on different sides of the bisector line of AB hence CD, $C'D'$ represent the two diagonals of the isosceles trapezoid $DC'CD'$).

We finally note that equality in (8) is attained for $ABCD$ degenerate with A, B, C, D collinear (and positioned in this order along the line on which they all lie). $\diamond$

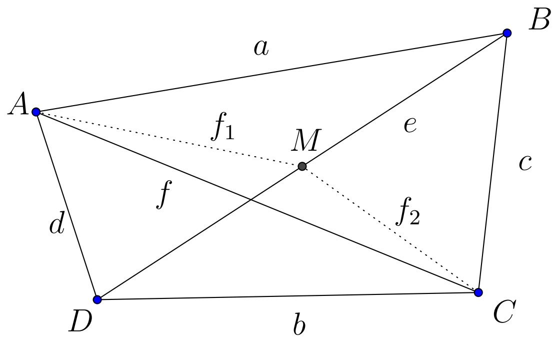
Problem 3

Let $ABCD$ be a quadrilateral whose side and diagonal lengths are $a \geq b \geq c \geq d$ and e, f respectively. Show that

$$e + f \leq a + b + (\sqrt{2} - 1)c + (\sqrt{2} - 1)d. \tag{9}$$

Solution

Let M be the midpoint of BD and denote by f_1 and f_2 the lengths of AM and CM as shown in the figure below.



By the well-known formula expressing the length of a median line in a triangle in terms of the three triangle side lengths (in turn a consequence of the cosine law) we have

$$e^2 + 4f_1^2 = 2(a^2 + d^2) \quad \text{and} \quad e^2 + 4f_2^2 = 2(b^2 + c^2)$$

from which we immediately obtain

$$(e + 2f_1)^2 \leq 4(a^2 + d^2) \quad \text{and} \quad (e + 2f_2)^2 \leq 4(b^2 + c^2).$$

By the triangle inequality and the estimates above we deduce

$$e + f \leq e + f_1 + f_2 \leq \sqrt{a^2 + d^2} + \sqrt{b^2 + c^2}$$

so that making use of $a \geq d$, $b \geq c$ too we obtain

$$\begin{aligned} e + f &\leq a + b + \sqrt{a^2 + d^2} - a + \sqrt{b^2 + c^2} - b \\ &= a + b + \frac{d^2}{\sqrt{a^2 + d^2} + a} + \frac{c^2}{\sqrt{b^2 + c^2} + b} \\ &\leq a + b + \frac{d^2}{\sqrt{d^2 + d^2} + d} + \frac{c^2}{\sqrt{c^2 + c^2} + c} \\ &= a + b + (\sqrt{2} - 1)c + (\sqrt{2} - 1)d. \end{aligned}$$

The proof is complete but we still note that equality is attained exactly for $ABCD$ square.