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d-COMPLETE SEQUENCES OF INTEGERS

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ABSTRACT. An infinite sequence $a_1 < a_2 < \cdots$ is *d-complete* if every sufficiently large integer is the sum of a_i such that no one divides the other. We investigate d-completeness of sets of the form $\{p^\alpha q^\beta\}$ and $\{p^\alpha q^\beta r^\gamma\}$ with α, β, γ nonnegative.

1. INTRODUCTION

An infinite sequence of integers $a_1 < a_2 < \cdots$ is called complete if every sufficiently large integer is the sum of distinct a_i. If every sufficiently large integer is the sum of a_i such that no one divides the other, we shall say that the sequence is d-complete.

In [1] Birch shows that the set $\{p^\alpha q^\beta\}$ with p and q relatively prime and α and β positive integers is complete. Cassels [2] considerably generalizes the result of Birch.

In this paper we are concerned with d-completeness of sets of the form $\{p^\alpha q^\beta\}$ and $\{p^\alpha q^\beta r^\gamma\}$ with α, β, γ nonnegative. This was motivated by a question asked by Paul Erdős: "*Is it true that every integer* > 1 *is the sum of distinct integers of the form* $2^\alpha 3^\beta$ (α *and* β *nonnegative integers*) *where no summand divides the other*?" Overestimating the difficulty of the problem, he told it to Jansen and wrote it to Lewin. Jansen almost immediately gave a simple proof by induction, which was also found by Lewin and by several others to whom Erdős wrote or told the problem. For the sake of completeness we shall reproduce the simple proof in the current paper. See also [3].

2. THE MAIN RESULTS

Proposition 1 (Appeared also as a "Quickie" in [3]). *The sequence* $\{2^\alpha 3^\beta\}$ *is d-complete.*

Proof. Up to $n = 3$ the proposition clearly holds.

Now let all the integers up to n, $3^p < n < 3^{p+1}$ of some p, be representable. If $n = 2m$, then m is representable by assumption and so is n. We may therefore assume n to be odd. But then $n - 3^p = 2m$, with $m < 3^p$, and so m is representable and 3^p does not divide any summand representing m. Then n is representable. This proves the proposition. □

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The question whether for n large enough it can always be written in the form $n = a_1 + a_2 + \cdots + a_k$ where all the a's are of the form $2^\alpha 3^\beta$ and all are in an interval $(x, 2x)$ yields a negative answer, since the number of integers of the form $2^\alpha 3^\beta$ in $(x, 2x)$ is asymptotically $\log x/\log 3$, so the number of subset sums is only about $x^{\log 2/\log 3}$, which is not enough to cover everything, even if each subset sum is different.

However, the following question may be asked. Is there a positive number t, for which the following holds: There exists a positive integer n_0 such that whenever $n > n_0$, then n can be expressed in the form $n = a_1 + a_2 + \cdots + a_k$, where all the a's are of the form $2^\alpha 3^\beta$ and all are in an interval (x, tx)?

If the answer to this question is affirmative, then one can ask how small t may be chosen.

We shall now show that Proposition 1 does not hold if we replace 3 (in Proposition 1) by 5. In fact we shall prove more.

Theorem 1. *Let p, q be coprime integers exceeding 1. If the positive integer s is not representable as a sum of members of the set $\{p^\alpha q^\beta\}$ with no summand dividing another, then neither are ps and qs.*

Proof. By symmetry it suffices to prove the theorem for ps. If ps is representable, say $ps = \sum p^{\alpha_i} q^{\beta_i}$, then at most one of the terms can have $\alpha_i = 0$. Since p and q are coprime, it follows that there cannot be exactly one term with $\alpha_i = 0$, for the left-hand side would be $\equiv 0 \pmod p$, whereas the right-hand side would be $q^{\beta_i} \not\equiv 0 \pmod p$. Thus $\alpha_i > 0$ in each term. Then $s = \sum p^{\alpha_i - 1} q^{\beta_i}$ is representable, a contradiction. □

Corollary. *For positive integers p and q, $\{p^\alpha q^\beta\}$ is d-complete if and only if $\{p, q\} = \{2, 3\}$.*

We now consider sequences whose terms are based on three primes (or powers of primes).

THE MAIN RESULTS

Let p be a prime greater than 5. We shall say that an integer n is p-representable if $n = \sum 2^\alpha 5^\beta p^\gamma$ for some nonnegative integers α, β, γ with no summand dividing another. We now have the following.

Proposition 2. *Every integer greater than* 34 *is* 11*-representable.*

Proof. Let n be the smallest integer exceeding 34 that is not 11-representable. We check the proposition by inspection for numbers up to 193 and find it true. Thus $n > 193$. If $n = 2m$ is even, then m is 11-representable and hence so is n. Thus we may assume n to be odd. We have $193 = 5^3 + 68$ and for $\alpha \geq 3$ we have $5^\alpha + 68 < 5^{\alpha-1} \times 11 + 68 < 3 \times 5^\alpha < 5^{\alpha+1} + 68 < 3 \times 5^{\alpha-1} \times 11$. It follows that for $\alpha \geq 3$, every $n > 193$ belongs to at least one of the two intervals $(5^\alpha + 68, 3 \times 5^\alpha)$ and $(5^{\alpha-1} \times 11 + 68, 3 \times 5^{\alpha-1} \times 11)$. We thus have

Case 1. $5^\alpha + 68 < n < 3 \times 5^\alpha$. Then $n = 5^\alpha + 2m$ with $34 < m < 5^\alpha$, and so m is 11-representable and hence so is n.

Case 2. $5^{\alpha-1} \times 11 + 68 < n < 3 \times 5^{\alpha-1} \times 11$. Then $n = 5^{\alpha-1} \times 11 + 2m$ with $34 < m < 5^{\alpha-1} \times 11$, and so m is 11-representable and hence so is n. This proves the proposition. □

We can show that for all primes p between 6 and 20 any integer greater than 155 is p-representable. Let $f(p)$ denote the largest integer which is not p-representable. Then $f(7) = 31$, $f(11) = 34$, $f(13) = 24$, $f(17) = 115$ and $f(19) = 155$.

We shall supply a proof for $p = 19$.

Proposition 3. *Every integer greater than* 155 *is* 19*-representable.*

Proof. Assume that $n > 155$ and that every integer between 155 and n is representable. Let α be a positive integer and consider the following strict inequality:

$$5^{\alpha+3} + 310 < 5^{\alpha} \times 19^2 + 310 < 5^{\alpha+4} + 310.$$

We check all the integers up to 935 by inspection, so that $n > 935$. As in the previous example, we may assume n odd. First assume that $n > 2115$. We now have

Case 1. $5^{\alpha+3} + 310 < n \leq 5^{\alpha} \times 19^2 + 310$. Put $n = 5^{\alpha+3} + 2m$. Then $310 < 2m \leq 236 \times 5^{\alpha} + 310$, and so $155 < m \leq 118 \times 5^{\alpha} + 155$. But since $n > 2115 = 5 \times 19^2 + 310$, it follows that $\alpha \leq 2$. Thus $118 \times 5^{\alpha} + 155 < 5^{\alpha+3}$, so that not only is m representable, but all of the summands used to represent m are less than $5^{\alpha+3}$. Hence n is representable.

Case 2. $19^2 \times 5^{\alpha} + 310 < n \leq 5^{\alpha+4} + 310$. Put $n = 19^2 \times 5^{\alpha} + 2m$, so that $155 < m \leq 132 \times 5^{\alpha} + 155$. Note that for any $\alpha \geq 0$ we have $132 \times 5^{\alpha} + 155 < 19^2 \times 5^{\alpha}$, so that not only is m representable, but all of the summands used to represent m are less than $19^2 \times 5^{\alpha}$. Hence n is representable.

It remains to consider numbers n in the interval $(935, 2115)$. First suppose $935 < n < 1875$. Write $n = 5^4 + 2m$, so that $155 < m < 5^4$. Thus not only is m representable, but all of the summands used to represent m are less than 5^4. Hence n is representable. Finally, suppose n is in the interval $(1875, 2115)$. Then $n = 5^4 + 19 \times 5^2 + 19^2 + 2m$, where $207 \leq m \leq 327$. Once again, not only is m representable, but all of the summands used to represent m are less than $19^2 \times 2$. Thus n is representable. This concludes the proof of Proposition 3. □

The cases $p = 7, 13$ and 17 are treated likewise.

We may sum up by stating

Theorem 2. *The sequence* $\{2^{\alpha}5^{\beta}p^{\gamma}\}$ *is d-complete for every prime p,* $6 < p < 20$.

As for primes greater than 20, it seems that our method does not always work satisfactorily. Already for $p = 23$ there seems to arise a difficulty stemming from the very small distance of the numbers 23 and 25. In any case we have not gone beyond 20.

Proposition 4. *The sequence* $\{3^{\alpha}, 5^{\beta}, 7^{\gamma}\}$ *is d-complete.*

Proof. We shall show that every integer exceeding 185 is representable.

We check all integers up to 2500. Let $N > 2500$ and let the theorem hold for all integers n, $185 < n < N$. Considering the fact that both $7^2/25$ and $25^2/7^3$ are numbers between one and two, we may choose for N an integer $N' = 25^{\alpha} \times 7^{\beta}$, where α, β are nonnegative integers such that $N/4 < N' < N/2$. We may also choose an integer $N'' = 5 \times 25^{\gamma} \times 7^{\delta}$ in the same interval. We now have

Case 1. $N \equiv 0 \pmod 3$. Then $N/3 > 185$ and hence is representable by assumption. Then N is representable.

Case 2. $N \equiv 1 \pmod 3$. Choose N' as described above. Then $N_1 = N - N' \equiv 0 \pmod 3$, with $N/2 < N_1 < (3/4)N$, and so $N = N' + 3 \times (N_1/3)$, where

$N_1/3 > N/6 > 185$, so that $N_1/3$ is representable. But $N' \equiv 1 \pmod 3$; the summands of N_1 being all divisible by 3, we get a representation of N without the summands dividing each other.

Case 3. $N \equiv 2 \pmod 3$. In this case we work with N'' and proceed as in Case 2. This proves the proposition. □

The following conjecture is perhaps true: *Let a, b, c be three integers which are pairwise relatively prime. Then every sufficiently large integer is d-representable by numbers of the form $a^\alpha b^\beta c^\gamma$.*

More generally, perhaps every sufficiently large n can be represented in the form $a_1 + a_2 + \cdots + a_k$, where $a_k \leq 2a_1$ and the a's are all of the form $a^\alpha b^\beta c^\gamma$.

Additional questions might be: If p and q are coprime and not 2 and 3, so that $\{p^\alpha, q^\beta\}$ is not d-complete, what can be said about the density of the nonrepresentable numbers? Are there infinitely many coprime nonrepresentables?

Perhaps the simplest conjecture with which we have difficulties states as follows:

Conjecture. For every t, there is an $n_0(t)$, such that every $n > n_0(t)$ can be represented as a sum of integers of the form $2^\alpha 3^\beta$, all of which are greater than t and none of which divides the other.

We can settle the conjecture, if we can prove that for every t there is an n, such that every integer between 3^{n-1} and 3^n can be represented as a sum of integers of the form $2^\alpha 3^\beta$, all of which are greater than t and none of which divides the other.

We are confident that for every fixed t we can find such an n by means of lengthy computations, but we do not see how to give a general proof.

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